

Monochromatic Hamiltonian Berge-cycles in colored hypergraphs

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Abstract

It has been conjectured that for any fixed r and sufficiently large n , there is a monochromatic Hamiltonian Berge-cycle in every $(r-1)$ -coloring of the edges of K_n^r , the complete r -uniform hypergraph on n vertices. In this paper, we show that the statement of this conjecture is true with $r-2$ colors (instead of $r-1$ colors) by showing that there is a monochromatic Hamiltonian t -tight Berge-cycle in every $\lfloor \frac{r-2}{t-1} \rfloor$ -edge coloring of K_n^r for any fixed $r > t \geq 2$ and sufficiently large n . Also, we give a proof for this conjecture when $r = 4$ (the first open case). These results improve the previously known results in [2, 3, 4].

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1 Introduction

For given $r \geq t \geq 2$, an r -uniform t -tight Berge-cycle of length n , denoted by $C_n^{(r,t)}$, is an r -uniform hypergraph with the core sequence v_1, v_2, \dots, v_n as the vertices, and distinct edges e_1, e_2, \dots, e_n such that e_i contains $v_i, v_{i+1}, \dots, v_{i+t-1}$ where addition is done modulo n . A t -tight Berge-cycle of length n in a hypergraph with n vertices is called a *Hamiltonian t -tight Berge-cycle*. This concept was introduced in [2] to generalize Berge-cycles ($t = 2$, [1]) and tight cycles ($t = r$, [8, 12]). Note that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a t -tight Berge-cycle $C_n^{(r,t)}$ is not determined uniquely and is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

Let H be an arbitrary r -uniform hypergraph. The *Ramsey number* $R_k(H)$ is the minimum integer n such that there is a monochromatic copy of H in every k -edge coloring of K_n^r . The existence of such a positive integer is guaranteed by Ramsey's classical result in [11]. Recently, the Ramsey numbers of various variations of cycles in uniform hypergraphs have been studied, e.g. see [7, 8, 10]. Considering this problem for Berge-cycles Gyárfás et al. proposed the following conjecture:

Conjecture 1. [3] *Assume that $r \geq 2$ is fixed and n is sufficiently large. Then every $(r-1)$ -edge coloring of K_n^r contains a monochromatic Hamiltonian Berge-cycle.*

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This conjecture states that for a given $r \geq 2$, $R_{r-1}(C_n^{(r,2)}) = n$ for sufficiently large n . Generalizing Conjecture 1 for t -tight Berge-cycles, Dorbec et al. proposed the following conjecture and they proved that if this conjecture is true it is best possible.

Conjecture 2. [2] *Assume that $c \geq 2$, $2 \leq t \leq r$, $c + t \leq r + 1$ and n is sufficiently large. Then every c -edge coloring of K_n^r contains a monochromatic Hamiltonian t -tight Berge-cycle.*

For general cases: It is proved that the statement of Conjecture 2 is true if we consider $ct + 1 \leq r$ instead of $c + t \leq r + 1$ see [2]. In [3] the authors proved a weaker form of Conjecture 1, which indicates that the statement of this conjecture is true for sufficiently large n with $\lfloor \frac{r-1}{2} \rfloor$ colors instead of $r - 1$ colors. In [6] the asymptotic form of Conjecture 1 was proved for every r using the method of Regularity Lemma. In fact, with the same assumptions the authors showed that there is a monochromatic Berge-cycle of length $(1 - o(1))n$ instead of a monochromatic Hamiltonian Berge-cycle. In this paper, we improve the first two results by showing that for any fixed $r > t \geq 2$ and sufficiently large n , there is a monochromatic Hamiltonian t -tight Berge-cycle in every $\lfloor \frac{r-2}{t-1} \rfloor$ -edge coloring of K_n^r . Clearly, this result implies that Conjecture 1 is true with $r - 2$ colors (instead of $r - 1$ colors).

For small cases: The case $c = 2$, $t = 3$ and $r = 4$ of Conjecture 2 was proved in [5]. In [3] Conjecture 1 was proved for $r = 3$ and an asymptotic result on this conjecture for $r = 4$ was obtained using the method of Regularity Lemma. Regarding the latter case, Gyárfás et al. [4], recently showed that for $n \geq 140$, in every 3-edge coloring of K_n^4 there is a monochromatic Berge-cycle of length at least $n - 10$. In the last section, we give a proof of Conjecture 1 for $r = 4$. Our proof involves new ideas (though, it modifies certain ideas from [4] at some points).

2 Monochromatic Hamiltonian t -tight Berge-cycles in colored hypergraphs

In this section, we show that there is a monochromatic Hamiltonian t -tight Berge-cycle in every $\lfloor \frac{r-2}{t-1} \rfloor$ -edge coloring of K_n^r for any fixed $r > t \geq 2$ with $r \geq 3$ and sufficiently large n . This establishes the statement of Conjecture 1 for $r - 2$ colors (instead of $r - 1$ colors) and improves the former known results in [2, 3]. In order to prove our result, we need some new definitions.

Assume that H is an r -uniform hypergraph. For a given cyclic order of $V(H)$, by a *consecutive t -vertices* we mean a subset of $V(H)$ consisting t consecutive elements. The *shadow t -graph* $\Gamma_t(H)$ is a t -uniform hypergraph (or t -graph) with vertex set $V(H)$, where the edges are the sets each consisting t distinct vertices for which there is an edge of H containing these vertices. Let $G = \Gamma_t(H)$ and c be a given l -edge coloring of H with colors $1, 2, \dots, l$. For each edge $e = x_1 x_2 \dots x_t$ of G , we assign a list $c(e)$ of colors of all edges of H containing x_1, x_2, \dots, x_t . For an edge e of G , the color $i \in c(e)$ is called *t -good* if at least $r - t + 1$ edges (of H) of color i contain all vertices of e . We consider G with a new multi-coloring c_t^* where $c_t^*(e) \subseteq c(e)$ is the set of all t -good colors for $e \in E(G)$. For $t = 2$, $l = 3$ and $H = K_n^4$, Gyárfás et al. showed that if there is a monochromatic Hamiltonian cycle C in G under multi-coloring c_2^* , then there is a monochromatic Hamiltonian Berge-cycle in H under edge coloring c (see Lemma 1 in [4]). Using the same argument, we give a generalization of their result as follows:

Lemma 2.1. *Let $r > t \geq 2$, c be a given l -edge coloring of $H = K_n^r$ and $G = \Gamma_t(H)$. Assume that there is a monochromatic Hamiltonian tight cycle in G under multi-coloring c_t^* . Then there is a monochromatic Hamiltonian t -tight Berge-cycle in H under c .*

Proof: Assume that C is a Hamiltonian tight cycle in G of color 1 (under c_t^*) with the core sequence x_1, x_2, \dots, x_n as the vertices. Then, following the cyclic order of vertices on C , suppose that A_j is the set of the edges of H in color 1 containing $x_j, x_{j+1}, \dots, x_{j+t-1}$. Since each A_j has at least $r-t+1$ elements and no element of A_j covers more than $r-t+1$ edges of C , Hall's theorem ensures the existence of a one-to one correspondence between all edges of C (all consecutive t -vertices of $V(C)$) and the sets A_j . This clearly defines a Hamiltonian t -tight Berge-cycle in H under coloring c . ■

Theorem 2.2. *Suppose that $r > t \geq 2$ and $n \geq (r-1)\lfloor \frac{r-2}{t-1} \rfloor + 2$. Then in every $\lfloor \frac{r-2}{t-1} \rfloor$ -edge coloring of K_n^r there is a monochromatic Hamiltonian t -tight Berge-cycle.*

Proof: Suppose to the contrary that there is no monochromatic Hamiltonian t -tight Berge-cycle in a given $\lfloor \frac{r-2}{t-1} \rfloor$ -edge coloring of K_n^r . For each $1 \leq i \leq \lfloor \frac{r-2}{t-1} \rfloor$, let S_i be the set of all edges e of $G = \Gamma_t(K_n^r)$ for which $i \notin c_t^*(e)$. Using Lemma 2.1, we may assume that the subhypergraph induced by $E(G) \setminus S_i$ in G does not have a Hamiltonian tight cycle.

Claim 2.3. *There are $(t-1)\lfloor \frac{r-2}{t-1} \rfloor + 1$ vertices in G so that the induced subhypergraph on these vertices in G and S_i have non-empty intersection, for each $1 \leq i \leq \lfloor \frac{r-2}{t-1} \rfloor$.*

We show by induction that for each $1 \leq l \leq \lfloor \frac{r-2}{t-1} \rfloor$ and any $\{S_{i_j}\}_{j=1}^l$ with $1 \leq i_j \leq \lfloor \frac{r-2}{t-1} \rfloor$ there are $(t-1)l + 1$ vertices in G so that for each $1 \leq j \leq l$ the edges of the induced subhypergraph on these vertices in G and S_{i_j} have non-empty intersection. The case $l = 1$ is trivial. Now assume that this holds for every $l < k$ where $k \leq \lfloor \frac{r-2}{t-1} \rfloor$. We verify case when $l = k$. First assume that for some $1 \leq s, t \leq k$ with $s \neq t$ there are two edges $e_s \in S_{i_s}$ and $e_t \in S_{i_t}$ with $|e_s \cap e_t| \geq 2$. By induction hypothesis there are $(t-1)(k-2) + 1$ vertices in G so that for each $1 \leq j \leq k$ and $j \neq s, t$, the edges of the induced subhypergraph on these vertices in G and S_{i_j} have non-empty intersection. By adding the vertices of $e_s \cup e_t$ to these $(t-1)(k-2) + 1$ vertices we get at most $(t-1)k + 1$ vertices with the desired property. So we may assume that $|e_s \cap e_t| \leq 1$ for any $1 \leq s, t \leq k$ with $s \neq t$ and any two edges $e_s \in S_{i_s}$ and $e_t \in S_{i_t}$. For each $1 \leq i \leq \lfloor \frac{r-2}{t-1} \rfloor$, let G_i be the subhypergraph of G induced by S_i and T_i be the set of all isolated vertices of G_i . Assume that H_i is the subhypergraph of G_i induced by $V(G_i) \setminus T_i$. We show that $\chi(H_i) > |T_i|$ for each $i \in \{i_1, \dots, i_k\}$. Assume to the contrary that for some $i \in \{i_1, \dots, i_k\}$ we have $\chi(H_i) \leq |T_i|$ and $C_1, C_2, \dots, C_{\chi(H_i)}$ are the color classes of H_i . Let $T_i = \{t_1, t_2, \dots, t_{|T_i|}\}$ and $V(C_j) = \{x_{j1}, x_{j2}, \dots, x_{jl_j}\}$ for $1 \leq j \leq \chi(H_i)$. Consider a cyclic order of vertices of G as follows:

$$S = \{t_1, x_{11}, \dots, x_{1l_1}, t_2, x_{21}, \dots, x_{2l_2}, \dots, x_{\chi(H_i)l_{\chi(H_i)}}, t_{\chi(H_i)+1}, t_{\chi(H_i)+2}, \dots, t_{|T_i|}\}.$$

Clearly each edge of G containing an element of T_i (also, each t -subset of any color class of H_i) is in $E(G) \setminus S_i$. Therefore, the set of all consecutive t -vertices in the cyclic order of vertices on S makes a Hamiltonian tight cycle for the subhypergraph induced by $E(G) \setminus S_i$ in G , a contradiction. So $\chi(H_i) > |T_i|$ for every $i \in \{i_1, \dots, i_k\}$. Clearly for each $1 \leq j \leq k$

and for any two color classes C_s and C_t with $s < t$ of H_{i_j} , there is an edge $e_{jst} \subseteq C_s \cup C_t$ in S_{i_j} . For such an edge e_{jst} assume that $A_{jst} = (e_{jst} \cap C_s) \times (e_{jst} \cap C_t)$. By the previous argument, $A_{jst} \cap A_{j's't'} \neq \emptyset$ if and only if $j = j', s = s'$ and $t = t'$. On the other hand, $|A_{jst}| \geq t - 1$. Therefore,

$$\sum_{j=1}^k \binom{|T_{i_j}|}{2} < (t-1) \sum_{j=1}^k \binom{|T_{i_j}|+1}{2} \leq \sum_{j=1}^k \sum_{1 \leq s < t \leq \chi(H_{i_j})} |A_{jst}|,$$

which means that there is an element $(u, v) \in A_{qst}$ for some q, s, t so that $\{u, v\} \not\subseteq T_{i_j}$ for each $1 \leq j \leq k$. Hence, for every $j \neq q$, there is an edge e_{i_j} in S_{i_j} containing at least one of u and v as a vertex. Therefore for each $1 \leq p \leq k$, the induced subhypergraph on $W = e_{qst} \cup \bigcup_{j \neq q} e_{i_j}$ and S_{i_p} have non-empty intersection. Clearly, $|W| \leq (t-1)k + 1$ which completes the proof of our claim.

Now, for every $1 \leq i \leq \lfloor \frac{r-2}{t-1} \rfloor$ let E_i be the set of all edges of color i in K_n^r containing all $(t-1)\lfloor \frac{r-2}{t-1} \rfloor + 1$ vertices disrupted in Claim 2.3. Clearly,

$$\sum_{i=1}^{\lfloor \frac{r-2}{t-1} \rfloor} |E_i| \geq n - (t-1)\lfloor \frac{r-2}{t-1} \rfloor - 1 \geq (r-t)\lfloor \frac{r-2}{t-1} \rfloor + 1.$$

On the other hand, for each $1 \leq i \leq \lfloor \frac{r-2}{t-1} \rfloor$ all edges in E_i contain the element $e_i \in S_i$ as a subset and $i \notin c_t^*(e_i)$. Hence, $|E_i| \leq r - t$ and so $\sum_{i=1}^{\lfloor \frac{r-2}{t-1} \rfloor} |E_i| \leq (r-t)\lfloor \frac{r-2}{t-1} \rfloor$, a contradiction. \blacksquare

The following interesting result on Conjecture 1 is an immediate consequence of Theorem 2.2 for $t = 2$.

Theorem 2.4. *Suppose that $n \geq r^2 - 3r + 4$. Then in every $(r-2)$ -edge coloring of K_n^r there is a monochromatic Hamiltonian Berge-cycle.*

3 Monochromatic Hamiltonian Berge-cycles in colored complete 4-graphs

Regarding the case $r = 4$ of Conjecture 1, an asymptotic result has been obtained using the method of Regularity Lemma; see [3]. Also, Gyárfás et al. [4], recently showed that for $n \geq 140$, in every 3-edge coloring of K_n^4 there is a monochromatic Berge-cycle of length at least $n - 10$. Here, we give a proof of Conjecture 1 for $r = 4$.

Lemma 3.1. *Suppose that $n \geq 85$ and the edges of $H = K_n^4$ are colored with three colors 1, 2, 3. If there exists a vertex $v \in V(H)$ such that for some $i \in \{1, 2, 3\}$, at most one edge of color i contains v , then there is a monochromatic Hamiltonian Berge-cycle in H .*

Proof: Assume that c is a 3-edge coloring of H where all edges containing $v_1 = v$ are colored with colors 2 and 3 except possibly the edge $e_{v_1} = \{v_1, v_2, v_3, v_4\}$. Without any loss of generality, we may assume that $c(e_{v_1}) \neq 3$. Consider c' as the new edge coloring of H such that $c'(e_{v_1}) = 2$ and $c'(e) = c(e)$ for any $e \in E(H) \setminus \{e_{v_1}\}$. A new 2-edge coloring for the 3-uniform complete hypergraph K with $n - 1$ vertices $V(H) \setminus \{v_1\}$ is induced by c'

as follows: The edge $\{x, y, z\}$ is of color 2 (resp. 3) in K if and only if the edge $\{v_1, x, y, z\}$ is of color 2 (resp. 3) under c' in H . By Theorem 1.2 in [3] there exists a monochromatic Hamiltonian Berge-cycle in K , say C . Let x_1, x_2, \dots, x_{n-1} be the core sequence of C . We consider the following cases:

Case 1. C is of color 3.

Without any loss of generality, we may assume that $x_1 \notin \{v_2, v_3, v_4\}$. If for some non-consecutive vertices x_k and $x_{k'}$ in $V(C) \setminus \{x_{n-1}, x_2\}$ the edge $\{x_1, x_k, x_{k'}\}$ is of color 3, then the cyclic order $x_1, v_1, x_2, \dots, x_{n-1}$ represents a core sequence of a Hamiltonian Berge-cycle of color 3 in H . It suffices to add v_1 to the edge $\{x_1, x_k, x_{k'}\}$ and all edges of C to get the edges of a Hamiltonian Berge-cycle of color 3 in H .

If $\{x_{n-1}, x_1, x_l\}$ (resp. $\{x_1, x_2, x_l\}$) is of color 3, for at least two numbers $l \neq n-2, 2$ (resp. $l \neq n-1, 3$), then we have the cyclic order v_1, x_1, \dots, x_{n-1} (resp. $x_1, v_1, x_2, \dots, x_{n-1}$) representing a core sequence of a Hamiltonian Berge-cycle of color 3 in H . It is sufficient to add v_1 to the edge $\{x_{n-1}, x_1, x_l\} \notin E(C)$ (resp. $\{x_1, x_2, x_l\} \notin E(C)$) and all edges of C to get the edges of a Hamiltonian Berge-cycle of color 3 in H .

Now, we may assume that $\{x_{n-1}, x_1, x_l\}$ (resp. $\{x_1, x_2, x_l\}$) is of color 2, for at least $n-6$ numbers $l \neq n-2, 2$ (resp. $l \neq n-1, 3$). Also, for any two non-consecutive vertices x_k and $x_{k'}$ in $V(C) \setminus \{x_{n-1}, x_2\}$, the edge $\{x_1, x_k, x_{k'}\}$ is of color 2. Consider a new cyclic order $y_1 = v_1, y_2, \dots, y_{n-1}, y_n = x_1$ for $V(H)$ such that for each $2 \leq i \leq n-1$, two vertices y_i and y_{i+1} don't appear as consecutive vertices in $V(C)$ and for any $2 \leq i \leq n-2$, the edge $\{y_i, y_{i+1}, x_1\}$ is of color 2. This is possible if we set $y_3 = x_{n-1}, y_6 = x_2$ and we choose y_2 and y_4 (also y_5 and y_7) as two non-consecutive vertices in $V(C) \setminus \{x_{n-2}, x_{n-1}, x_1, x_2, x_3\}$ such that $\{y_3, y_i, x_1\}$ for $i = 2, 4$ and $\{y_6, y_i, x_1\}$ for $i = 5, 7$ are of color 2. The cyclic order y_1, y_2, \dots, y_n defines a Hamiltonian Berge-cycle of color 2 in H with the following edge assignments. Set $e_i = \{v_1, y_i, y_{i+1}, x_1\}$ for $2 \leq i \leq n-2$, $e_{n-1} = \{v_1, y_p, y_{n-1}, x_1\}$, $e_n = \{v_1, y_h, y_k, x_1\}$ and $e_1 = \{v_1, y_2, y_l, x_1\}$, where y_p, y_h, y_k and y_l are pairwise non-consecutive vertices in $V(C) \setminus \{y_{n-2}, y_{n-1}, y_n, y_1, y_2, y_3, y_6\}$ and y_{n-1} and y_p (also, y_2 and y_l) are non-consecutive vertices in $V(C)$.

Case 2. C is of color 2.

If $\{v_2, v_3, v_4\} \notin E(C)$, then by an argument similar to that in case 1 we can see that there is a monochromatic Hamiltonian Berge-cycle in H . Now, suppose that the edge $e_1 = \{v_2, v_3, v_4\}$ appears in $E(C)$ to cover the consecutive vertices v_2 and v_3 . We may assume that $x_1 = v_2, x_2 = v_3, x_3, \dots, x_{n-1}$ is the core sequence of the cycle C where for each $1 \leq i \leq n-1$, $e_i \in E(C)$ is the edge containing x_i and x_{i+1} . If there are two distinct edges $\{v_2, x_k, x_{k'}\}$ and $\{v_3, x_l, x_{l'}\}$ of color 2 in $E(K) \setminus E(C)$, then we consider the cyclic order of vertices of H as $y_1 = v_2, y_2 = v_1, y_3 = v_3, y_4 = x_3, \dots, y_n = x_{n-1}$. The edges $f_1 = \{v_2, v_1, x_k, x_{k'}\}$, $f_2 = \{v_1, v_3, x_l, x_{l'}\}$ and for $3 \leq i \leq n$, $f_i = e_{i-1} \cup \{v_1\}$ define a Hamiltonian Berge-cycle of color 2 in H . So we may assume that for at least one of the vertices v_2 and v_3 , say v_2 , all the edges $\{v_2, x_k, x_{k'}\} \neq e_1, e_{n-1}$ are of color 3 where x_k and $x_{k'}$ are non-consecutive vertices of C . Now, we consider a new cyclic order $y_1 = v_1, y_2, y_3, \dots, y_{n-1}, y_n = v_2$ of the vertices $V(H)$, where for any $2 \leq i \leq n-1$, y_i, y_{i+1} are not consecutive vertices in $V(C)$ and for any $2 \leq i \leq n-2$ the edge $\{y_i, y_{i+1}, v_2\}$

is of color 3. Clearly v_3 and v_4 are not consecutive vertices of the mentioned cyclic order. The following edge assignments for this cyclic order represent a Hamiltonian Berge-cycle of color 3 in H , which completes the proof. Set $f_i = \{v_1, y_i, y_{i+1}, v_2\}$ for $2 \leq i \leq n-2$, $f_{n-1} = \{v_1, y_p, y_{n-1}, v_2\}$, $f_n = \{v_1, y_h, y_k, v_2\}$ and $f_1 = \{v_1, y_2, y_l, v_2\}$, where $4 \leq p, h, k, l \leq n-3$, y_p, y_h, y_k and y_l are non-consecutive vertices in $V(C) \setminus (\{v_3, v_4\} \cup e_{n-1})$ and y_{n-1} and y_p (also, y_2 and y_l) are non-consecutive vertices in $V(C)$. ■

Theorem 3.2. *Any 3-edge coloring of K_n^4 with $n \geq 85$ contains a monochromatic Hamiltonian Berge-cycle.*

Proof: Assume that c is a 3-edge coloring of $H = K_n^4$ with colors 1, 2, 3. In [4] under the same assumptions Gyárfás et al. showed that if $|c_2^*(e)| = 1$ for an edge e of $G = \Gamma_2(H)$, then there is a monochromatic Hamiltonian Berge-cycle in H . So suppose that for any edge e of G , we have $|c_2^*(e)| \geq 2$.

Let v be an arbitrary vertex. Define $U_{12}(v)$, $U_{13}(v)$, $U_{23}(v)$ and $U_{123}(v)$ as the sets to which v is connected (in the multi-coloring c_2^*) in color sets 12, 13, 23 and 123, respectively. For $i, j, k \in \{1, 2, 3\}$ in some order, define

$$B_i = \{v \in V(G) \mid U_{ij}(v) = U_{ik}(v) = \emptyset, U_{jk}(v) \neq \emptyset\}, B_4 = \{v \in V(G) \mid |U_{123}(v)| \geq \frac{n}{2}\}.$$

It is easy to see that for $i \neq 4$, B_i 's are pairwise disjoint and for an edge e of G from B_i to B_j where $i, j \in \{1, 2, 3\}$ and $i \neq j$, we have $c_2^*(e) = \{1, 2, 3\}$. Also, in [4] it has been shown that if $V(G) = \bigcup_{i=1}^4 B_i$, then there is a monochromatic Hamiltonian cycle for G under the multi-coloring c_2^* and so by Lemma 2.1 for $r = 4$ and $t = 2$, we conclude that there is a monochromatic Hamiltonian Berge-cycle in H . So suppose that $\bigcup_{i=1}^4 B_i \neq V(G)$. For every $v \in V(G) \setminus \bigcup_{i=1}^4 B_i$, consider $\pi(v) = \min\{|U_{23}(v)|, |U_{12}(v)|, |U_{13}(v)|\}$. We choose a vertex $x \in V(G) \setminus \bigcup_{i=1}^4 B_i$ with minimum $|U_{123}(x)|$, among those with minimum $\pi(x)$. In the sequel, for simplicity we denote $U_{ij}(x)$ and $U_{123}(x)$ ($i, j \in \{1, 2, 3\}$) by U_{ij} and U_{123} , respectively. Let $U = V(G) \setminus (\{x\} \cup U_{123})$ and without any loss of generality, assume that $|U_{23}| \leq |U_{12}| \leq |U_{13}|$. One can easily see that $U_{12} \neq \emptyset$ and $|U| \geq \lfloor \frac{n}{2} \rfloor$.

In [4] it has been shown that $|U_{23}| \leq 1$. Now we show that if $|U_{23}| = 1$, then $|U_{12}| \leq 2$. Since $|U| \geq \lfloor \frac{n}{2} \rfloor$, there are at least nine vertices in U_{13} . Let S be a subset of U_{13} of cardinality 9. Suppose that $u \in U_{23}$ and $T \subseteq U_{12}$ with $|T| = 3$. There are twenty seven edges in H each consisting x, u , one of the vertices in T and one member of S . On the other hand, at most two of these edges are of color 1 (each edge has u as a vertex), at most six of them are of color 3 (each edge has exactly one of the vertices in T) and at most eighteen of them are of color 2 (each edge has exactly one of the vertices in S), a contradiction.

In the sequel, we assume that $y \in U_{12}$ and $z \in U_{13}$ are fixed vertices and we define a Hamiltonian graph Γ with $V(\Gamma) = V(H)$, in such a way that every Hamiltonian cycle C of Γ can be extended to a monochromatic Hamiltonian Berge-cycle in H . For this, we consider the following two cases:

Case 1. Let $|U_{23}| = \emptyset$.

Let U_{123} and U_{12} be partitioned into A, B and A', B' respectively, where $|B| \leq |A| \leq |B| + 1$ and $|B'| \leq |A'| \leq |B'| + 1$. Suppose that $y \in A'$. Consider a graph Γ with the vertex set $V(\Gamma) = V(H)$ and the edge set $E(\Gamma) = \bigcup_{i=1}^7 E_i$, where E_i s are defined as follows:

- $E_1 = \{uv | u \in U_{12} \setminus \{y\}, v \neq y, c(\{x, z, u, v\}) = 1\}$.
- $E_2 = \{uv | u \in U_{13} \setminus \{z\}, c(\{x, y, u, v\}) = 1\}$.
- $E_3 = \{zv | v \in V(\Gamma) \setminus A \cup A', c(\{x, y, z, v\}) = 1\}$.
- $E_4 = \{yv | v \in A \cup A', c(\{x, y, z, v\}) = 1\}$.
- $E_5 = \{yv | v \in U_{13} \setminus \{z\}\}$.
- Assume that $U_{123} = \{w_1, w_2, \dots, w_m\}$ and $d_{\Gamma'}(w_1) \leq d_{\Gamma'}(w_2) \leq \dots \leq d_{\Gamma'}(w_m)$, where Γ' is the graph induced by $\bigcup_{i=1}^5 E_i$. For $i = 1, 2$, assume that $e_{w_i v} = \{x, z, w_i, v\}$ when $w_i v \in E_1 \cup E_4$ and $e_{w_i v} = \{x, y, w_i, v\}$ when $w_i v \in E_2 \cup E_3$. Since $1 \in c_2^*(xw_i)$ there are $r_i = \max\{3 - d_{\Gamma'}(w_i), 0\}$ edges $W_i = \{g_{i1}, \dots, g_{ir_i}\} \subseteq E(H) \setminus \{e_{w_i v} | w_i v \in \bigcup_{i=1}^4 E_i\}$ of color 1 containing x and w_i for $i = 1, 2$. Consider the following cases:

- i. $r_1 \leq 2$. Set $E'_6 = D_{w_1} = \emptyset$. If $r_2 \leq 1$, then set $E''_6 = D_{w_2} = D'_{w_2} = \emptyset$. Now assume that $r_2 = 2$. Let $W'_2 = \{g_{21}, g_{22}, e_{w_2 v}\}$ where $w_2 v \in \bigcup_{i=1}^4 E_i$. Set $D'_{w_2} = \emptyset$, $E''_6 = \{w_2 t_1\}$ and $D_{w_2} = g_{21} \setminus \{x, w_2, t_1\}$ where $t_1 \in g_{21} \setminus \{x, w_2, v\}$. Now let $r_2 = 3$. Set $E''_6 = \{w_2 t_1, w_2 t_2\}$, $D_{w_2} = g_{21} \setminus \{x, w_2, t_1\}$ and $D'_{w_2} = g_{22} \setminus \{x, w_2, t_2\}$ where $t_1 \in g_{21} \setminus \{x, w_2\}$ and $t_2 \in g_{22} \setminus \{x, w_2, t_1\}$ are the vertices with maximum repetitions in g_{21} and g_{22} .
- ii. $r_1 = 3$. So we have $W_1 = \{g_{11}, g_{12}, g_{13}\}$. If $r_2 \leq 1$, then set $E''_6 = D_{w_2} = D'_{w_2} = \emptyset$, $E'_6 = \{w_1 u\}$ and $D_{w_1} = g_{11} \setminus \{x, w_1, u\}$ where $u \in g_{11} \setminus \{x, w_1\}$. If $r_2 = 2$, then $W_2 = \{g_{21}, g_{22}\}$. Let $W'_2 = \{g_{21}, g_{22}, e_{w_2 v}\}$ where $w_2 v \in \bigcup_{i=1}^4 E_i$. We may assume that $g_{13} \notin W'_2$. Set $E'_6 = \{w_1 u\}$, $E''_6 = \{w_2 t_1\}$, $D_{w_1} = g_{13} \setminus \{x, w_1, u\}$, $D_{w_2} = g_{21} \setminus \{x, w_2, t_1\}$ and $D'_{w_2} = \emptyset$ so that $u \in g_{13} \setminus \{x, w_1\}$ and $t_1 \in g_{21} \setminus \{x, w_2, v\}$. Now let $r_2 = 3$. If $W_1 \cap W_2 = \emptyset$, then set $E'_6 = \{w_1 u\}$, $E''_6 = \{w_2 t_1, w_2 t_2\}$, $D_{w_1} = g_{11} \setminus \{x, w_1, u\}$, $D_{w_2} = g_{21} \setminus \{x, w_2, t_1\}$ and $D'_{w_2} = g_{22} \setminus \{x, w_2, t_2\}$ so that $u \in g_{11} \setminus \{x, w_1\}$, $t_1 \in g_{21} \setminus \{x, w_2\}$, $t_2 \in g_{22} \setminus \{x, w_2, t_1\}$ and t_1 and t_2 have maximum repetitions in g_{21} and g_{22} . Otherwise, we may assume that $|g_{1i} \cap \{w_2\}| \geq |g_{1j} \cap \{w_2\}|$ for $i < j$. Choose $t_1 \in g_{22} \setminus \{x, w_1, w_2\}$ and set $E'_6 = \{w_1 w_2\}$, $E''_6 = \{w_2 w_1, w_2 t_1\}$, $D_{w_1} = g_{11} \setminus \{x, w_1, w_2\}$, $D_{w_2} = g_{22} \setminus \{x, w_1, w_2, t_1\}$ and $D'_{w_2} = \emptyset$.

In all cases set $E_6 = E'_6 \cup E''_6$ and $D = D_{w_1} \cup D_{w_2} \cup D'_{w_2}$.

- $E_7 = \{xv | v \in (V(\Gamma) \setminus (\{x, y, z\} \cup D)) \cup \{w_1, w_2\}\}$.

Claim 3.3. *The graph Γ is Hamiltonian.*

Assume that $d_1 \leq d_2 \leq \dots \leq d_n$ are degrees of the vertices of Γ . Now we show that for each $i \leq \frac{n}{2}$, we have $d_i > i$ or $d_{n-i} \geq n - i$. So Chvátal's condition [9] implies the existence of a Hamiltonian cycle in Γ . Clearly, $d_\Gamma(x) \geq n - 6$. When $u \in U_{12} \setminus \{y\}$, apart

from at most four choices of $v \in V(\Gamma) \setminus \{u, x, y, z\}$ the edges $\{x, z, u, v\}$ of H are of color 1. So $d_\Gamma(u) \geq n - 8$ where $u \in U_{12} \setminus \{y\}$. Similarly, $d_\Gamma(u) \geq n - 7$ for $u \in U_{13} \setminus \{z\}$ and also we have $d_\Gamma(u) \geq n - 6$ when $u \in U_{13} \setminus (\{z\} \cup D)$. It is straightforward to see that $d_\Gamma(z) \geq n - |A| - |A'| - 7 \geq \frac{n+1}{2}$ and $d_\Gamma(y) \geq n - |B| - |B'| - 7 \geq \frac{n+3}{2}$. For $U_{123} = \emptyset$, Chvátal's condition implies that the graph Γ is Hamiltonian. Now let $U_{123} = \{w_1, w_2, \dots, w_m\} \neq \emptyset$, $|U_{12} \setminus \{y\}| = l$, $|U_{13}| = k$ and suppose that $d_\Gamma(w_i) \leq d_\Gamma(w_{i+1})$ for every $1 \leq i \leq m - 1$. For $i = 1, \dots, m$, let

$$N_i = \{\{x, z, v, w_i\} | v \in U_{12} \setminus \{y\}\} \cup \{\{x, y, v, w_i\} | v \in U_{13}\}.$$

For each $1 \leq i \leq m$, suppose that n_i is the number of edges of color 1 in N_i . Clearly, $d_\Gamma(w_i) \geq n_i$, for each $1 \leq i \leq m$. Among all $m(k + l)$ edges in $\bigcup_{i=1}^m N_i$, there are at most $2(k + l) + 2$ edges of colors 2 and 3. So $\sum_{i=1}^m n_i \geq (m - 2)(k + l) - 2$. If $d_\Gamma(w_3) \leq \lfloor \frac{k+l}{3} \rfloor - 1$, then $\sum_{i=1}^3 n_i \leq \sum_{i=1}^3 d_\Gamma(w_i) \leq k + l - 3$. Therefore,

$$\sum_{i=4}^m n_i \geq (m - 2)(k + l) - 2 - (k + l - 3) = (m - 3)(k + l) + 1,$$

which is impossible, since $|\bigcup_{i=4}^m N_i| = (m - 3)(k + l)$. Thus, $d_\Gamma(w_3) \geq \lfloor \frac{k+l}{3} \rfloor > 9$ and consequently $d_\Gamma(w_i) \geq 10$ for $3 \leq i \leq 6$. On the other hand, by the definitions of E_6 and E_7 we have $d_\Gamma(w_1) \geq 2$ and $d_\Gamma(w_2) \geq 3$. Hence,

$$d_i > i \quad \text{for } 3 \leq i \leq 6. \quad (3.1)$$

Since $|U_{123}| < \frac{n}{2}$ and $|U_{13}| \geq \frac{1}{2} \lfloor \frac{n}{2} \rfloor$, we have $d_{n-i} \geq n - i$ for each $6 \leq i \leq \frac{n}{2}$. On the other hand, by (3.1), we have $d_i > i$ for $1 \leq i \leq 6$. Now clearly Chvátal's condition yields the existence of a Hamiltonian cycle in Γ .

Claim 3.4. *Every Hamiltonian cycle in Γ can be extended to a monochromatic Hamiltonian Berge-cycle of color 1 in H .*

Suppose that $v_1, v_2, \dots, v_{n-1}, v_n = x$ is the vertices of a Hamiltonian cycle C in Γ . Without any loss of generality, we may assume that $v_1 \neq w_1$. Now for $i = 1, 2, \dots, n$, we define the edges $f_i \in E(H)$ of color 1 in the same order their subscripts appear so that $\{v_i, v_{i+1}\} \subseteq f_i$ and f_1, f_2, \dots, f_n make a Hamiltonian Berge-cycle with the core sequence v_1, v_2, \dots, v_n . First let $i = 1, 2, \dots, n - 2$. Set $f_i = \{x, z, v_i, v_{i+1}\}$ for $v_i v_{i+1} \in E_1 \cup E_4$ and $f_i = \{x, y, v_i, v_{i+1}\}$ for $v_i v_{i+1} \in E_2 \cup E_3$. If $v_i v_{i+1} \in E_5$, then set $f_i = \{x, v_i, v_{i+1}, u\}$ of color 1 so that $u \in U_{13} \setminus \{z, v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_1, v_{n-1}\}$. Such an edge exists since $|U_{13}| \geq \frac{1}{2} \lfloor \frac{n}{2} \rfloor \geq 20$ and for a fixed vertex $v \in U_{13} \setminus \{z\}$ there are at least $q = \frac{1}{2} \lfloor \frac{n}{2} \rfloor - 6 > 14$ vertices, say $\{u_1, u_2, \dots, u_q\}$ in $U_{13} \setminus \{z, v\}$, where every edge $\{x, y, v, u_j\}$ is of color 1. If $v_i v_{i+1} \in E_6$, then by the definition of E_6 , there is an appropriate edge $f_i \in W_1 \cup W_2$ containing v_i and v_{i+1} . Now let $i = n - 1$. It is easy to see that $\{v_{n-1}, x\}$ has been used in at most two of the edges f_i s for $1 \leq i \leq n - 2$. On the other hand, $1 \in c_2^*(v_{n-1}x)$. Thus we can choose an appropriate edge f_{n-1} . Finally let $i = n$. One can see that $\{x, v_1\}$ has been used in at most two of the edges f_i s for $1 \leq i \leq n - 1$ and since $1 \in c_2^*(xv_1)$, then there is an appropriate edge f_n .

Case 2. $|U_{23}| = 1$.

Since $|U_{23}| = 1$, we have $1 \leq |U_{12}| \leq 2$. Assume that $U_{23} = \{u_{23}\}$, $U_{12} = \{y, u_{12}\}$ for $|U_{12}| = 2$ and $U_{12} = \{y\}$ for $|U_{12}| = 1$. If $\frac{n-2}{2} \leq |U_{123}| \leq \frac{n-1}{2}$ and $U_{123} \subseteq B_1$, then for each $w \in U_{123}$ and each $v \in V(G)$ we have $2 \in c_2^*(wv)$. On the other hand, $2 \in c_2^*(xu_{23})$ and $2 \in c_2^*(xy)$ and so Chvátal's condition implies that the subgraph induced by all edges e with $2 \in c_2^*(e)$ contains a Hamiltonian cycle. By Lemma 2.1 for $r = 4$ and $t = 2$, the proof is completed.

Now fix a vertex $w \in U_{123} \setminus B_1$ when $\frac{n-2}{2} \leq |U_{123}| \leq \frac{n-1}{2}$. Let U_{123} be partitioned into A, B , where $|B| \leq |A| \leq |B| + 1$. Consider a graph Γ with the vertex set $V(\Gamma) = V(H)$, and the edge set $E(\Gamma) = \bigcup_{i=1}^8 E_i$, where E_i s are defined as follows:

- $E_1 = \emptyset$ when $U_{12} = \{y\}$ and $E_1 = \{u_{12}v | v \neq y, c(\{x, z, u_{12}, v\}) = 1\}$, otherwise.
- $E_2 = \{uv | u \in U_{13} \setminus \{z\}, c(\{x, y, u, v\}) = 1\}$.
- $E_3 = \{zv | v \in V(\Gamma) \setminus A, c(\{x, y, z, v\}) = 1\}$.
- $E_4 = \{yv | v \in A, c(\{x, y, z, v\}) = 1\}$.
- $E_5 = \{yv | v \in U_{13} \setminus \{z\}\}$.
- Assume that $U_{123} = \{w_1, w_2, \dots, w_m\}$ and $d_{\Gamma'}(w_1) \leq d_{\Gamma'}(w_2) \leq \dots \leq d_{\Gamma'}(w_m)$, where Γ' is the graph induced by $\bigcup_{i=1}^5 E_i$. Assume that $e_{w_1v} = \{x, z, w_1, v\}$ (resp. $e_{u_{23}v} = \{x, z, u_{23}, v\}$) when $w_1v \in E_1 \cup E_4$ (resp. $u_{23}v \in E_1$) and $e_{w_1v} = \{x, y, w_1, v\}$ (resp. $e_{u_{23}v} = \{x, y, u_{23}, v\}$) when $w_1v \in E_2 \cup E_3$ (resp. $u_{23}v \in E_2 \cup E_3$). By Lemma 3.1 and the fact $1 \in c_2^*(xw_1)$, there are $r = \max\{3 - d_{\Gamma'}(w_1), 0\}$ and $l = \max\{2 - d_{\Gamma'}(u_{23}), 0\}$ edges $W = \{h_1, \dots, h_r\} \subseteq E(H) \setminus \{e_{w_1v} | w_1v \in \bigcup_{i=1}^4 E_i\}$ and $U = \{g_1, \dots, g_l\} \subseteq E(H) \setminus \{e_{u_{23}v} | u_{23}v \in \bigcup_{i=1}^3 E_i\}$ of color 1 containing $\{x, w_1\}$ and u_{23} , respectively. We consider three cases:

- i. $r \leq 1$. Set $E'_6 = D_{w_1} = D'_{w_1} = \emptyset$. If $l = 0$, then set $E''_6 = D_{u_{23}} = D'_{u_{23}} = \emptyset$. If $l = 1$, then $U' = \{g_1, e_{u_{23}v}\}$ where $u_{23}v \in \bigcup_{i=1}^3 E_i$. Set $D'_{u_{23}} = \emptyset$, $E'_6 = \{u_{23}t_1\}$, $D_{u_{23}} = g_1 \setminus \{x, u_{23}, t_1\}$ so that $t_1 \in g_1 \setminus \{x, u_{23}, v\}$. Now let $l = 2$. Thus $U = \{g_1, g_2\}$. Set $E''_6 = \{u_{23}t_1, u_{23}t_2\}$, $D_{u_{23}} = g_1 \setminus \{x, u_{23}, t_1\}$ and $D'_{u_{23}} = g_2 \setminus \{x, u_{23}, t_2\}$ so that $t_1 \in g_1 \setminus \{x, u_{23}\}$ and $t_2 \in g_2 \setminus \{x, u_{23}, t_1\}$ and t_1 and t_2 have maximum repetitions in g_1 and g_2 .
- ii. $r = 2$. Set $D'_{w_1} = \emptyset$. Let $W' = \{h_1, h_2, e_{w_1v}\}$ where $w_1v \in \bigcup_{i=1}^4 E_i$. If $l = 0$, then set $E''_6 = D_{u_{23}} = D'_{u_{23}} = \emptyset$, $E'_6 = \{w_1u_1\}$, $D_{w_1} = h_1 \setminus \{x, w_1, u_1\}$ where $u_1 \in h_1 \setminus \{x, w_1, v\}$. If $l = 1$, then $U' = \{g_1, e_{u_{23}u}\}$ where $u_{23}u \in \bigcup_{i=1}^3 E_i$. We may assume that $|h_1 \cap \{u_{23}\}| \geq |h_2 \cap \{u_{23}\}|$. Set $D'_{u_{23}} = \emptyset$, $E'_6 = \{w_1u_1\}$, $E''_6 = \{u_{23}t_1\}$, $D_{w_1} = h_2 \setminus \{x, w_1, u_1\}$, $D_{u_{23}} = g_1 \setminus \{x, u_{23}, t_1\}$ so that $u_1 \in h_2 \setminus \{x, w_1, v\}$, $t_1 \in g_1 \setminus \{x, u_{23}, u\}$. Now let $l = 2$. Then we have $U = \{g_1, g_2\}$. If $W' \cap U = \emptyset$, then set $E'_6 = \{w_1u_1\}$, $E''_6 = \{u_{23}t_1, u_{23}t_2\}$, $D_{w_1} = h_1 \setminus \{x, w_1, u_1\}$, $D_{u_{23}} = g_1 \setminus \{x, u_{23}, t_1\}$ and $D'_{u_{23}} = g_2 \setminus \{x, u_{23}, t_2\}$ so that $u_1 \in h_1 \setminus \{x, w_1, v\}$, $t_1 \in g_1 \setminus \{x, u_{23}\}$, $t_2 \in g_2 \setminus \{x, u_{23}, t_1\}$ and t_1 and t_2 have maximum repetitions in g_1 and g_2 . Otherwise, we may assume that $h_1 = g_1$. Set $D'_{u_{23}} = \emptyset$, $E'_6 = \{w_1u_{23}\}$, $E''_6 = \{u_{23}w_1, u_{23}t_1\}$, $D_{w_1} = h_1 \setminus \{x, w_1, u_{23}\}$, $D_{u_{23}} = g_2 \setminus \{x, u_{23}, t_1\}$ where $t_1 \in g_2 \setminus \{x, u_{23}, w_1\}$ and t_1 has maximum repetition in g_1 and g_2 .

iii. $r = 3$. If $l = 0$, then set $E'_6 = D_{u_{23}} = D'_{u_{23}} = \emptyset$, $E'_6 = \{w_1u_1, w_1u_2\}$, $D_{w_1} = h_1 \setminus \{x, w_1, u_1\}$, $D'_{w_1} = h_2 \setminus \{x, w_1, u_2\}$ where $u_1 \in h_1 \setminus \{x, w_1\}$, $u_2 \in h_2 \setminus \{x, w_1, u_1\}$ and u_1 and u_2 have maximum repetitions in h_1 and h_2 . If $l = 1$, then $U' = \{g_1, e_{u_{23}v}\}$ where $u_{23}v \in \bigcup_{i=1}^3 E_i$. We may assume that $g_1 \notin \{h_2, h_3\}$. Now set $D'_{u_{23}} = \emptyset$, $E'_6 = \{w_1u_1, w_1u_2\}$, $E''_6 = \{u_{23}t_1\}$, $D_{w_1} = h_2 \setminus \{x, w_1, u_1\}$, $D'_{w_1} = h_3 \setminus \{x, w_1, u_2\}$, $D_{u_{23}} = g_1 \setminus \{x, u_{23}, t_1\}$ where $u_1 \in h_2 \setminus \{x, w_1\}$, $u_2 \in h_3 \setminus \{x, w_1, u_1\}$, $t_1 \in g_1 \setminus \{x, u_{23}, v\}$ and u_1 and u_2 have maximum repetitions in h_2 and h_3 . Now let $l = 2$. If $W \cap U = \emptyset$, then set $E'_6 = \{w_1u_1, w_1u_2\}$, $E''_6 = \{u_{23}t_1, u_{23}t_2\}$, $D_{w_1} = h_1 \setminus \{x, w_1, u_1\}$, $D'_{w_1} = h_2 \setminus \{x, w_1, u_2\}$, $D_{u_{23}} = g_1 \setminus \{x, u_{23}, t_1\}$, $D'_{u_{23}} = g_2 \setminus \{x, u_{23}, t_2\}$ so that $u_1 \in h_1 \setminus \{x, w_1\}$, $u_2 \in h_2 \setminus \{x, w_1, u_1\}$ are vertices with maximum repetitions in h_1 and h_2 and $t_1 \in g_1 \setminus \{x, u_{23}\}$ and $t_2 \in g_2 \setminus \{x, u_{23}, t_1\}$ are vertices with maximum repetitions in g_1 and g_2 . Otherwise, we may assume that $|h_i \cap \{u_{23}\}| \geq |h_j \cap \{u_{23}\}|$ for $i < j$, $h_1 = g_1$ and $h_3 \notin U$. Set $D'_{u_{23}} = \emptyset$, $E'_6 = \{w_1u_{23}, w_1u_1\}$, $E''_6 = \{u_{23}w_1, u_{23}t_1\}$, $D_{w_1} = h_1 \setminus \{x, w_1, u_{23}\}$, $D'_{w_1} = h_3 \setminus \{x, w_1, u_1\}$, $D_{u_{23}} = g_2 \setminus \{x, u_{23}, w_1, t_1\}$ so that $u_1 \in h_3 \setminus \{x, w_1\}$, $t_1 \in g_2 \setminus \{x, u_{23}, w_1\}$ and t_1 has maximum repetition in g_1 and g_2 .

In all cases set $E_6 = E'_6 \cup E''_6$ and $D = D_{w_1} \cup D'_{w_1} \cup D_{u_{23}} \cup D'_{u_{23}}$.

- $E_7 = \emptyset$ if $|U_{123}| \leq \frac{n-3}{2}$ and $E_7 = \{wv \mid v \in V(\Gamma) \setminus \{x, w\}, 1 \in c_2^*(vw)\}$, otherwise. It is easy to see that $|E_7| \geq \frac{n}{2}$ when $\frac{n-2}{2} \leq |U_{123}| \leq \frac{n-1}{2}$.
- $E_8 = \{xv \mid v \in (V(\Gamma) \setminus (\{x, y, z, u_{23}, w\} \cup D)) \cup \{w_1\}\}$.

Claim 3.5. *The graph Γ is Hamiltonian.*

Assume that $d_1 \leq d_2 \leq \dots \leq d_n$ are degrees of the vertices of Γ . Now we show that for each $i \leq \frac{n}{2}$, we have $d_i > i$ or $d_{n-i} \geq n - i$. So Chvátal's condition implies the existence of a Hamiltonian cycle in Γ . According to the above discussions $d_\Gamma(x) \geq n - 11$. For every $u \in U_{13} \setminus \{z\}$, with at most four choices of $v \in V(\Gamma) \setminus \{x, y, u\}$ excluded the edges $\{x, y, u, v\}$ of H are of color 1. So $d_\Gamma(u) \geq n - 7$, where $u \in U_{13} \setminus \{z\}$ and also we have $d_\Gamma(u) \geq n - 6$ when $u \in U_{13} \setminus (\{z\} \cup D)$. Similarly, $d_\Gamma(u_{12}) \geq n - 8$ when $U_{12} = \{y, u_{12}\}$. It is straightforward to see that $d_\Gamma(u_{23}) \geq 2$, $d_\Gamma(z) \geq n - |A| - 7 \geq \frac{n+5}{2}$ and $d_\Gamma(y) \geq n - |B| - 9 \geq \frac{n+3}{2}$. If $U_{123} = \emptyset$, then one can easily see that Chvátal's condition implies that the graph Γ is Hamiltonian.

Now let $U_{123} = \{w_1, w_2, \dots, w_m\} \neq \emptyset$ with $d_\Gamma(w_1) \leq d_\Gamma(w_2) \leq \dots \leq d_\Gamma(w_m)$ and $|U_{13}| = k$. We show that $d_\Gamma(w_m) \geq \frac{n}{2}$ when $\frac{n-2}{2} \leq |U_{123}| \leq \frac{n-1}{2}$. If $w \in \bigcup_{i=2}^4 B_i$, then one can easily see that $|E_7| = |\{v \mid v \in V(\Gamma) \setminus \{x, w\}, 1 \in c_2^*(wv)\}| \geq \frac{n}{2}$. So $d_\Gamma(w_m) \geq d_\Gamma(w) \geq \frac{n}{2}$. Now let $w \in V(G) \setminus \bigcup_{i=1}^4 B_i$. From the definition of x , we conclude that $\frac{n-2}{2} \leq |U_{123}(w)| \leq \frac{n-1}{2}$ and $U_{13}(w)$ and $U_{12}(w)$ are non-empty. Hence again $|E_7| = |\{v \mid v \in V(\Gamma) \setminus \{x, w\}, 1 \in c_2^*(wv)\}| \geq \frac{n}{2}$ and so we have $d_\Gamma(w_m) \geq d_\Gamma(w) \geq \frac{n}{2}$ when $\frac{n-2}{2} \leq |U_{123}| \leq \frac{n-1}{2}$.

Now we claim that

$$d_i > i \quad \text{for } 1 \leq i \leq 6. \quad (3.2)$$

First let $U_{12} = \{y, u_{12}\}$ and let $T = \{\{x, y, u_{23}, v\}, \{x, u_{12}, u_{23}, v\} \mid v \in U_{13}\}$. Let S be the set of all vertices $v \in U_{13}$ for which there is an edge of color 1 or 3 containing v in T .

Clearly, $|S| \leq 6$. Therefore, for each $v \in U_{13} \setminus S$, since $2 \notin c_2^*(xv)$ apart from two edges in T all edges of H containing x and v are of color 1 or 3. On the other hand, since $3 \notin c_2^*(xy)$ at most two edges $\{x, y, v, w_i\}$ are of color 3 where $i \in \{1, 2, \dots, m\}$. So for each $i = 1, \dots, m$ at least $k - 8 > 30$ edges in $\{\{x, y, v, w_i\} | v \in U_{13} \setminus S\}$ are of color 1. Hence, $d_\Gamma(w_1) > 30$ and so $d_i > i$ for each $1 \leq i \leq 6$.

Now let $U_{12} = \{y\}$ and $T = \{\{x, y, u_{23}, v\} | v \in U_{13}\}$. At least $k-4$ edges in T are of color 2 in H . Now for $i = 1, \dots, m$, consider $N_i = \{\{x, y, v, w_i\} | v \in U_{13}\}$. For every $1 \leq i \leq m$, suppose that n_i is the number of edges of color 1 in N_i . Clearly, $d_\Gamma(w_i) \geq n_i$. Let $S_T \subseteq U_{13}$ be the set of all vertices v that lies on an edge of T of color 2. Clearly, $|S_T| \geq k-4$. Since $2 \notin c_2^*(xv)$ for each $v \in U_{13}$, there are at most $|S_T|$ (resp. $2(k - |S_T|)$) edges of color 2 in $\bigcup_{i=1}^m N_i$ each containing a vertex in S_T (resp. $U_{13} \setminus S_T$). Therefore, among all mk edges in $\bigcup_{i=1}^m N_i$ there are at most $k+6$ edges of colors 2 and 3. So $\sum_{i=1}^m n_i \geq (m-1)k - 6$. If $d_\Gamma(w_2) \leq \lfloor \frac{k-7}{2} \rfloor$, then $\sum_{i=1}^2 n_i \leq \sum_{i=1}^2 d_H(w_i) \leq k-7$. Therefore,

$$\sum_{i=3}^m n_i \geq (m-1)k - 6 - (k-7) = (m-2)k + 1,$$

which is impossible, since $|\bigcup_{i=3}^m N_i| = (m-2)k$. Thus, $d_\Gamma(w_2) > \lfloor \frac{k-7}{2} \rfloor \geq 15$ and consequently $d_\Gamma(w_i) \geq 16$ for $2 \leq i \leq 6$. On the other hand, according to the definitions of E_6 and E_8 , we have $d_{u_{23}} \geq 2$ and $d_{w_1} \geq 3$. Therefore, $d_i > i$ for each $1 \leq i \leq 6$.

Based on the previous discussions, since $|U_{123}| \leq \frac{n-1}{2}$ and $|U_{13}| \geq \lfloor \frac{n}{2} \rfloor - 3$, we have $d_{n-i} \geq n-i$ for each $6 \leq i \leq \frac{n}{2}$. On the other hand by (3.2), we have $d_i > i$ for $1 \leq i \leq 6$. Now, Chvátal's condition implies the existence of a Hamiltonian cycle in H .

Claim 3.6. *There is a Hamiltonian Berge-cycle of color 1 in H .*

We show that every Hamiltonian cycle in Γ can be extended to a monochromatic Hamiltonian Berge-cycle in H . Suppose that $v_1, v_2, \dots, v_{n-1}, v_n = x$ is the vertices of a Hamiltonian cycle C in Γ . Now for each $i = 1, 2, \dots, n$, we define an edge $f_i \in E(H)$ of color 1 in the same order their subscripts appear so that $\{v_i, v_{i+1}\} \subseteq f_i$ and f_1, f_2, \dots, f_n make a Hamiltonian Berge-cycle with the core sequence v_1, v_2, \dots, v_n . First let $i \in [n] \setminus (\{n-1, n\} \cup \{i | v_i v_{i+1} \in E_7\})$, where $[n] = \{1, 2, \dots, n\}$. Set $f_i = \{x, z, v_i, v_{i+1}\}$ for $v_i v_{i+1} \in E_1 \cup E_4$ and $f_i = \{x, y, v_i, v_{i+1}\}$ for $v_i v_{i+1} \in E_2 \cup E_3$. Now let $v_i v_{i+1} \in E_5$. Set $f_i = \{x, v_i, v_{i+1}, u\}$ of color 1, where $u \in U_{13} \setminus \{z, v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_1, v_{n-1}\}$. Such an edge exists since $|U_{13}| \geq \lfloor \frac{n}{2} \rfloor - 3$ and for a fixed vertex $v \in U_{13} \setminus \{z\}$ there are at least $q = \lfloor \frac{n}{2} \rfloor - 9 > 30$ vertices, say $\{u_1, u_2, \dots, u_q\}$ in $U_{13} \setminus \{z, v\}$, where every edge $\{x, y, v, u_j\}$ is of color 1. If $v_i v_{i+1} \in E_6$, then by the definition of E_6 , there is an appropriate edge $f_i \in W \cup U$ containing v_i and v_{i+1} .

Now let $L_{uv} \subset E(H) \setminus \{f_i | i \in [n] \setminus (\{n-1, n\} \cup \{i | v_i v_{i+1} \in E_7\})\}$ be the set of all edges of color 1 containing u and v . Note that $1 \in c_2^*(v_{n-1}x)$ and $1 \in c_2^*(xv_1)$. By the definitions of E_6 and E_7 , it is easy to see that Hall's theorem implies that we can choose appropriate edges $f_{n-1} \in L_{v_{n-1}x}$, $f_n \in L_{xv_1}$ and $f_i \in L_{v_i v_{i+1}}$ for each i with $v_i v_{i+1} \in E_7$. ■

References

- [1] C. Berge, Graphs and Hypergraphs, North Holland, Amsterdam and London, 1973.

- [2] P. Dorbec, S. Gravier and G.N. Sárközy, Monochromatic Hamiltonian t -tight Berge-cycles in hypergraphs, *J. Graph Theory* **59** (2008), 34–44.
- [3] A. Gyárfás, J. Lehel, G.N. Sárközy and R.H. Schelp, Monochromatic Hamiltonian Berge-cycles in colored complete uniform hypergraphs, *J. Combin. Theory Ser. B.* **98** (2008), 342–358.
- [4] A. Gyárfás, G.N. Sárközy and E. Szemerédi, Long monochromatic Berge-cycles in colored 4-uniform hypergraphs, *Graphs Combin.* **26** (2010), 71–76.
- [5] A. Gyárfás, G.N. Sárközy and E. Szemerédi, Monochromatic Hamiltonian 3-tight Berge cycles in 2-colored 4-uniform hypergraphs, *J. Graph Theory* **63** (2010), 288–299.
- [6] A. Gyárfás, G.N. Sárközy and E. Szemerédi, Monochromatic matchings in the shadow graph of almost complete hypergraphs, *Ann. Combin.* **14** (2010), 245–249.
- [7] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, M. Simonovits and J. Skokan, The Ramsey number for hypergraph cycles I, *J. Combin. Theory Ser. A* **113** (2006), 67–83.
- [8] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński and J. Skokan, The Ramsey number for 3-uniform tight hypergraph cycles, *Combin. Probab. Comput.* **18** (2009), 165–203.
- [9] L. Lovász, Combinatorial Problems and Exercises, 2nd edn. Akadémiai Kiadó, North Holland (1976).
- [10] G.R. Omid and M. Shahsiah, Ramsey numbers of 3-uniform loose paths and loose cycles, *J. Combin. Theory Ser. A*, **121** (2014), 64–73.
- [11] F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc. Ser. 2* **30** (1930), 264–286.
- [12] V. Rödl, A. Ruciński and E. Szemerédi, A dirac-type theorem for 3-uniform hypergraphs, *Comb. Probab. Comput.* **15** (2006), 229–251.